Genus and braid index associated to sequences of renormalizable Lorenz maps

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Abstract

We describe the Lorenz links generated by renormalizable Lorenz maps with reducible kneading invariant $(K_f^-, K_f^+) = (X, Y) * (S, W)$, in terms of the links corresponding to each factor. This gives one new kind of operation that permits us to generate new knots and links from old. Using this result we obtain explicit formulas for the genus and the braid index of this renormalizable Lorenz knots and links. Then we obtain explicit formulas for sequences of these invariants, associated to sequences of renormalizable Lorenz maps with kneading invariant $(X, Y) * (S, W)^{*n}$, concluding that both grow exponentially. This is specially relevant, since it is known that topological entropy is constant on the archipelagoes of renormalization.

1 Introduction

Let ϕ_t be a flow on S^3 with countably many periodic orbits $(\tau_n)_{n=1}^\infty$. We can look to each closed orbit as a knot in S^3. It was R. f. Williams, in 1976, who first conjectured that non trivial knotting occur in the Lorenz system ([13]). In 1983, Birmann and Williams introduced the notion of template, in order to study the knots and links (i.e. finite collections of knots, taking into account the knotting between them) contained in the geometric Lorenz attractor ([2]).

A template, or knot holder, consists of a branched two manifold with charts of two specific types, joining and splitting, together with an expanding semiflow defined on it, see Figure 1. The relationship between templates and links of periodic orbits in three dimensional flows is expressed in the following result, known as Template Theorem, due to Birman and Williams in [2].

Theorem 1 *Given a flow ϕ_t on a three-manifold M, having a hyperbolic chain-recurrent set, the link of periodic orbits L_ϕ is in bijective correspondence with the link of periodic orbits $L_\mathcal{T}$ on a particular embedded template $\mathcal{T} \subset M$. On any finite sublink, this correspondence is via ambient isotopy.*
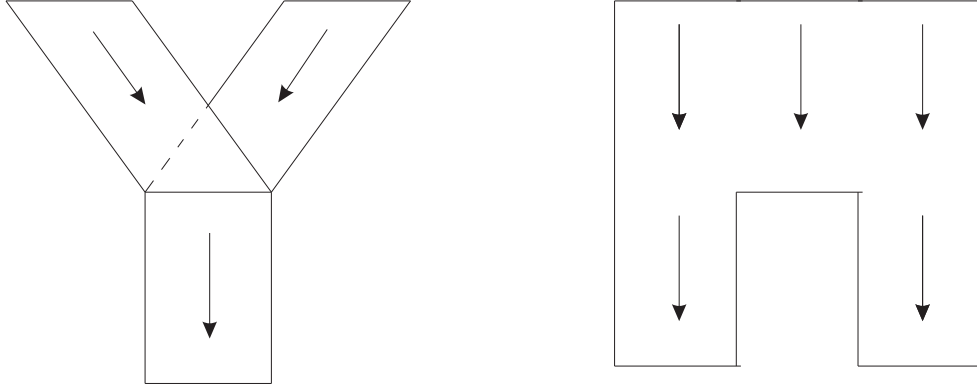
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Figure 1: Charts of templates: joining (left) and splitting (right)

The dynamics of the semiflow on the Lorenz template are described by the first-return map to the branch line, which consists of a one-dimensional map with one discontinuity, surjective and strictly increasing in each continuity subinterval.

What we now call a *Lorenz flow* has a singularity of saddle type with a one-dimensional unstable manifold and an infinite set of hyperbolic periodic orbits, whose closure contains the saddle point (see [10]). To describe the dynamics of such a flow it is necessary to add a geometric hypotheses, just like the one introduced in [14] to study the original Lorenz system. A Lorenz flow with this extra assumption is called a *Geometric Lorenz flow*. The dynamics of all these flows can be described by first-return one-dimensional maps with one discontinuity, that are not necessarily surjective in the continuity subintervals. This maps are called Lorenz maps, more precisely, we will adopt the following definition introduced in [10].

Definition 1 *Let $P < 0 < Q$ and $r \geq 1$. A C^r Lorenz map $f : [P,Q] \to [P,Q]$ is a map described by a pair (f_-, f_+) where:*

1. *$f_- : [P,0] \to [P,Q]$ and $f_+ : [0,Q] \to [P,Q]$ are continuous and strictly increasing maps;*

2. *$f(P) = P$, $f(Q) = Q$ and f has no other fixed points in $[P,Q]\backslash\{0\}$.*

3. *There exists $\rho > 0$, the exponent of f, such that*

$$f_-(x) = \widetilde{f}_-(|x|^\rho) \ \ and \ \ f_+(x) = \widetilde{f}_+(|x|^\rho)$$

where $\widetilde{f}_-$ and $\widetilde{f}_+$, the coefficients of the Lorenz map, are C^r diffeomorphisms defined on appropriate closed intervals.

Because of the ambiguity at the point 0, we consider the map undefined in 0. This Lorenz map is denoted by (P,Q,f_-,f_+) (if there is no ambiguity about the interval of definition, we erase the corresponding symbols P,Q).

In [10], Martens and de Melo introduced some parametrized families of Lorenz maps that are universal in the sense that, given any geometric Lorenz flow, its dynamics is essentially the same as the dynamics of some element of the family, more precisely, consider $\mathcal{L}^r$ the collection of all Lorenz maps of class C^r. Endow

$\mathcal{L}^r$ with a topology that takes care of the domain, of the exponents and of the coefficients.

Definition 2 *Let $\Lambda \subset \mathbb{R}^2$ be closed. A Lorenz family is a continuous map $F : \Lambda \to \mathcal{L}^r$,*

$$F_\lambda = (P_\lambda, Q_\lambda, \varphi_\lambda, \psi_\lambda).$$

A *monotone Lorenz family* is a C^3 Lorenz family such that:

1. F_λ has negative Schwarzian derivative for all $\lambda \in \Lambda$;

2. $\Lambda = [0,1] \times [0,1]$;

3. $F : (s,t) \to (-1, 1, \varphi_s, \psi_t)$ and $\rho_{s,t} = \rho > 1$;

4. If $s_1 < s_2$ then $\varphi_{s_1}(x) < \varphi_{s_2}(x)$ for all $x \in [-1, 0]$ and if $t_1 < t_2$ then $\psi_{t_1}(x) < \psi_{t_2}(x)$ for all $x \in [0, 1]$;

5. $\varphi_0(0) = 0, \varphi_1(0) = 1, \psi_0(0) = -1$ and $\psi_1(0) = 0$;

6. $DF_\lambda(\pm 1) > 1$ for all $\lambda \in \Lambda$.

In [10] it is proved that Monotone Lorenz families are full, in the sense that, if F_λ is a monotone Lorenz family, then for each given C^2 Lorenz map f there is a parameter λ such that the dynamics of F_λ are essentially the same as the dynamics of f.

In [6], Holmes studied families of iterated horseshoe knots which arise naturally associated to sequences of period-doubling bifurcations of unimodal maps.

It is well known, see for example [3], that period doubling bifurcations in the unimodal family are directly related with the creation of a 2-renormalization interval, i.e. a subinterval $J \subset I$ containing the critical point, such that $f^2|_J$ is unimodal.

Basically there are two types of bifurcations in Lorenz maps (see [11]): the usual saddle-node or tangent bifurcations, when the graph of f^n is tangent to the diagonal $y = x$, and one attractive and one repulsive n-periodic orbits are created or destroyed; homoclinic bifurcations, when $f^n(0^{\pm}) = f^{n-1}(f_{\pm}(0)) = 0$ and one attractive n-periodic orbit is created or destroyed in this way, these bifurcations are directly related with homoclinic bifurcations of flows modelled by this kind of maps (see [11]).

Considering a monotone family of Lorenz maps, the homoclinic bifurcations are realized in some lines in the parameters space, called *hom-lines* or *bifurcation bones*.

It is known that (see [11] and [9]), in the context of Lorenz maps, renormalization intervals are created in each intersection of two hom-lines. These points are called *homoclinic points* and are responsible for the self-similar structure of the bifurcation skeleton of monotone families of Lorenz maps. So it is reasonable to say that *homoclinic points are the Lorenz version of period-doubling bifurcation points.*

The idea of symbolic dynamics is to associate to each orbit of a map, a symbolic sequence, called the *itinerary* of the corresponding point under the map. The pairs of sequences corresponding to the orbits of the critical points determine all the combinatorics of the map and are called *kneading invariants*; from this point of view, the renormalizability of a map is equivalent to the reducibility of its kneading invariant as the $*$-product of two other kneading invariants (see below for the complete definitions).

At the topological and dynamical levels, we know a lot about the structure of renormalizable Lorenz maps, but, from the point of view of knots and links generated by these maps, as far as we know, this question was only superficially approached in [12]. So the objective of this work is to describe the structure and invariants of knots and links generated by renormalizable Lorenz maps with kneading invariants of type $(X, Y) * (S, T)$, by means of the ones generated by (X, Y) and (S, T). Then we will study sequences of invariants of knots and links generated by sequences of kneading invariants corresponding to the iteration of the $*$-product, i.e., corresponding to pairs of type

$$(X, Y) * (K^-, K^+)^{*n} = (X(n), Y(n)),$$

where $(K^-, K^+)^{*n}$ denotes the $*$-product of (K^-, K^+) with itself $n-1$ times.

Our main theorem describes the links corresponding to n-tuples of (renormalizable) periodic itineraries of type $((X, Y) * Z_1, \ldots, (X, Y) * Z_n)$ in terms of the links corresponding to (X, Y) and $(Z_1, \ldots, Z_n)$. This gives one kind of operation that permits us to generate new knots and links from old. Note that, unlike the case of Horseshoe knots, studied by Holmes in [6], this operation do not corresponds to cabling or any other operation that we know. Using this result we proceed obtaining explicit formulas for the genus and the braid index of knots and links corresponding to reducible sequences or pairs of sequences depending, respectively, on the genus and on the braid index of each factor. Then we obtain explicit formulas for these invariants associated to pairs of type $(X, Y) * (K^-, K^+)^{*n} = (X(n), Y(n))$, concluding that both the genus and the braid index grow exponentially through these sequences. This is specially relevant since it is known (see [9]) that the topological entropy is constant in the renormalization archipelagoes, and each of these sequences is contained in one archipelago. So in this cases knot theory provides much finer invariants for the classification of flows.

2   Symbolic dynamics of Lorenz maps

Denoting by $f^j = f \circ f^{j-1}$, $f^0 = id$, the j-th iterate of the map f, we define the *itinerary* of a point x under a Lorenz map f as $i_f(x) = (i_f(x))_j, j = 0, 1, \ldots$, where

$$(i_f(x))_j = \begin{cases} L & \text{if} \quad f^j(x) < 0 \\ 0 & \text{if} \quad f^j(x) = 0 \\ R & \text{if} \quad f^j(x) > 0 \end{cases}.$$

It is obvious that the itinerary of a point x will be a finite sequence in the symbols L and R with 0 as its last symbol, if and only if x is a pre-image of 0 and otherwise it is one infinite sequence in the symbols L and R. So it is natural to consider the symbolic space Σ of sequences $X_0 \cdots X_n$ on the symbols $\{L, 0, R\}$, such that $X_i \neq 0$ for all $i < n$ and: $n = \infty$ or $X_n = 0$, with the lexicographic order relation induced by $L < 0 < R$.

It is straightforward to verify that, for all $x, y \in [-1, 1]$, we have

1. If $x < y$ then $i_f(x) \leq i_f(y)$, and

2. If $i_f(x) < i_f(y)$ then $x < y$.

We define the *kneading invariant* associated to a Lorenz map $f = (f_-, f_+)$, as

$$K_f = (K_f^-, K_f^+) = (Li_f(f_-(0)), Ri_f(f_+(0))).$$

We say that a pair $(X, Y) \in \Sigma \times \Sigma$ is *admissible* if $(X, Y) = K_f$ for some Lorenz map f.

Consider the *shift map* $s : \Sigma \setminus \{0\} \to \Sigma$, $s(X_0 \cdots X_n) = X_1 \cdots X_n$. The set of admissible pairs is characterized, combinatorially, in the following way (see, for example, [9]).

Proposition 1 *A pair $(X, Y) \in \Sigma \times \Sigma$ is admissible if and only if $X_0 = L$, $Y_0 = R$ and, for $Z \in \{X, Y\}$ we have:*
(1) If $Z_i = L$ then $s^i(Z) \leq X$;
(2) If $Z_i = R$ then $s^i(Z) \geq Y$; with inequality (1) (resp. (2)) strict if X (resp. Y) is finite.

A sequence $X \in \Sigma$ is said to be $f - admissible$ if there exists $x \in [-1, 1]$ such that $i_f(x) = X$. The f-admissible sequences are completely determined by the kneading invariant K_f (see for example [9]), i. e., a sequence X is f-admissible if and only if it verifies the following conditions:

1. If $X_i = L$ then $s^i(X) \leq K_f^-$;

2. If $X_i = R$ then $s^i(X) \geq K_f^+$; with strict inequalities in the finite cases.

2.1 Renormalization and $*$-product

In the context of Lorenz maps, we define renormalizability on the following way, see for example [11]:

Definition 3 *Let f be a Lorenz map, then we say that f is (n, m) renormalizable if there exist points $P < y_L < 0 < y_R < Q$ such that*

$$g(x) = \begin{cases} f^n(x) & if \ y_L \leq x < 0 \\ f^m(x) & if \ 0 < x \leq y_R \end{cases}$$

is a Lorenz map.

The map $R_{(n,m)}(f) = g = (f^n, f^m)$ is called the (n, m)-renormalization of f and $[y_L, y_R]$ is the corresponding renormalization interval.

A sequence $X \in \Sigma$ is said to be *maximal* if $X_0 = L$ and $s^i(X) \leq X$ for all i such that $X_i = L$, analogously a sequence $Y \in \Sigma$ is *minimal* if $Y_0 = R$ and $s^i(Y) \geq Y$ for all i such that $Y_i = R$.

It is easy to prove that one infinite periodic sequence $(X_0 \cdots X_{m-1})^\infty$ with least period m (the exponent ∞ denotes the indefinite repetition of the sequence), is maximal (resp. minimal) if and only if the finite sequence $X_0 \cdots X_{m-1}0$ is maximal (resp. minimal).

Let $|X|$ be the length of a finite sequence $X = X_0 \cdots X_{|X|-1}0$, from the last observation it is reasonable to identify each finite maximal or minimal sequence $X_0 \cdots X_{|X|-1}0$ with the corresponding infinite periodic sequence $(X_0 \cdots X_{|X|-1})^\infty$, this is the case, for example, when we talk about the knot associated to a finite sequence.

It is also easy to prove that a pair of finite sequences

$$(X_0 \ldots X_{|X|-1}0, Y_0 \ldots Y_{|Y|-1}0)$$

is admissible, if and only if the pair of infinite periodic sequences

$$((X_0 \cdots X_{|X|-1})^\infty, (Y_0 \cdots Y_{|Y|-1})^\infty)$$

is admissible.

Considering a monotone family of Lorenz maps, F_λ, the homoclinic bifurcations are realized in the lines in the parameters space such that the finite sequence X is realized as the left element of the kneading invariant, if X is maximal and as the right element if X is minimal. These lines are called *hom-lines* or *bifurcation bones* and can be defined as

$$B(X) = \{\lambda \in \Lambda : K^-_{F_\lambda} = X\}.$$

if X is maximal and

$$B(Y) = \{\lambda \in \Lambda : K^+_{F_\lambda} = Y\}$$

if Y is minimal.

The union of the bifurcation bones is usually called the *bifurcation skeleton* (Figure 2.1).
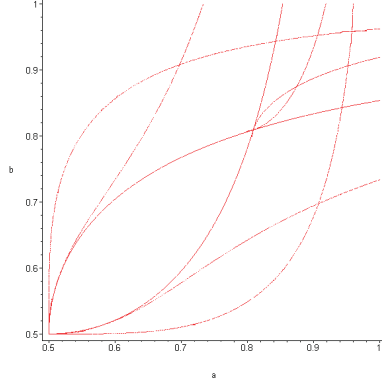

Figure 2: Part of the bifurcation skeleton, namely, the bones with end point on the right side are the maximal bones corresponding to, from down to up, $LRL0$, $LR0$, $LRRL0 = (LR0, RL0) * LR0$ and $LRR0$. With end point on the top we have the minimal bones corresponding, from left to right, to $RLL0$, $RL0$, $RLLR0 = (LR0, RL0) * RL0$ and $RLR0$. The intersection of each two lines is the "vertex" of a similar copy of the all picture.

Obviously, two maximal or minimal bones corresponding to different sequences can never intersect, so the only intersections we have in the bifurcation skeleton are between maximal and minimal bones. These points are called homoclinic and are where renormalization intervals are created.

We define the $*$-product between a pair of finite sequences $(X, Y) \in \Sigma \times \Sigma$, and a sequence $U \in \Sigma$ as

$$(X, Y) * U = \overline{U}_0 \overline{U}_1 \cdots \overline{U}_{|U|-1}0,$$

where

$$\overline{U}_i = \begin{cases} X_0 \cdots X_{|X|-1} & if\ U_i = L \\ Y_0 \cdots Y_{|Y|-1} & if\ U_i = R \end{cases}.$$

Now we define the $*$-product between two pairs of sequences, $(X,Y),(U,T) \in \Sigma \times \Sigma$, X and Y finite, as

$$(X,Y) * (U,T) = ((X,Y) * U, (X,Y) * T).$$

The next theorem states that the reducibility relative to the $*$-product is equivalent to the renormalizability of the map. The proof can be found, for example, in [9].

Theorem 2 *Let f be a Lorenz map, then f is (n,m)-renormalizable iff there exist two admissible pairs (X,Y) and (U,T) such that $|X| = n$, $|Y| = m$, $K_f = (X,Y) * (U,T)$ and $K_{R_{(n,m)}(f)} = (U,T)$.*

We also know from [9] that the product $(X,Y) * (U,T)$ is admissible if and only if both (X,Y) and (U,T) are admissible, so for each finite admissible pair (X,Y), the subspace $(X,Y) * \{all\ admissible\ pairs\}$ is isomorphic to the all space $\{all\ admissible\ pairs\}$, this provides a self-similar structure in the symbolic space of kneading invariants and, correspondingly, in the bifurcation skeleton. At the topological and dynamical levels, the structure of the maps in these similar subspaces is well described by renormalization and $*$-product, but, from the point of view of knots and links generated by these maps, as far as we know, this question was only superficially approached in [12]. So the objective of this work is to describe the structure and invariants of knots and links generated by Lorenz maps with kneading invariants of type $(X,Y) * (S,T)$, relating it with the ones generated by (X,Y) and (S,T).

First we will state some useful properties of the $*$-product.

Proposition 2 *Let (X,Y) be one admissible pair of finite sequences, and $Z < Z'$, then $(X,Y) * Z < (X,Y) * Z'$.*

The proof is straightforward.

Denote by $X^\infty = (X_0 \ldots X_{|X|-1})^\infty = (X,Y)*L^\infty$ and by $Y^\infty = (Y_0 \ldots Y_{|Y|-1})^\infty = (X,Y) * R^\infty$. The previous Proposition implies that $X^\infty \leq (X,Y) * Z \leq Y^\infty$ for any sequence Z.

Remark 1 *From now on, we will freely identify the finite sequence $X_0 \ldots X_{|X|-1}0$, with the periodic sequence $X^\infty = (X_0 \ldots X_{|X|-1})^\infty$, wherever it is convenient one or the other interpretation, for example, if $p > |X|$ then we use X_p to denote the element $X_{p \mod |X|}$.*

Lemma 1 *Let (X,Y) be one admissible pair of finite sequences, $0 < q < |Y|$ and $Y_q = R$, then $Y_q \cdots Y_{|Y|-1}(X,Y) * Z \geq Y^\infty$, for any sequence Z. Analogously, if $0 < q < |X|$ and $X_q = L$, then $X_q \cdots X_{|X|-1}(X,Y) * Z \leq X^\infty$, for any sequence Z.*

Proof

Since (X,Y) is admissible, then $Y_q \ldots Y_{|Y|-1}Y^\infty > Y^\infty$, so there exists l such that $Y_q \cdots Y_{q+l-1} = Y_0 \ldots Y_{l-1}$ and $Y_{q+l} > Y_l$. If $q + l < |Y|$ the result follows immediately. If $q + l \geq |Y|$, then necessarily $Y_{|Y|-q} = L$, because otherwise we

would have $Y^\infty > Y_{|Y|-q}\dots$ and $Y_{|Y|-q} = R$, and this violates admissibility. But then,

$$Y_{|Y|-q}\cdots Y_{|Y|-1}Y^\infty \leq X^\infty \leq (X,Y) * Z \tag{1}$$

and this gives the result. The proof of the second part is analogous. ∎

Remark 2 *Note that, equality in the first inequality of (1), implies that $s^{|Y|-q}(Y^\infty) = X^\infty$ so, if this is not the case, then the inequalities in the previous Lemma are strict.*

Proposition 3 *Let (X,Y) be one admissible pair of finite sequences and $W, W' \in \{X,Y\}$. If $s^p(W^\infty) < s^q(W'^\infty)$ and $W_p\cdots W_{|W|-1} \neq W'_q\cdots W'_{|W'|-1}$ then*

$$W_p\cdots W_{|W|-1}(X,Y) * Z \leq W'_q\cdots W'_{|W'|-1}(X,Y) * Z'$$

for any sequences Z, Z'.

Proof The proof is divided in four cases: $W = X$ and $W' = Y$; $W = Y$ and $W' = X$; $W = W' = X$ and $W = W' = Y$. We will only demonstrate specifically the first case, since the others follow with analogous arguments..

Following the hypotheses, there exists l such that $X_p\cdots X_{p+l-1} = Y_q\cdots Y_{q+l-1}$ and $X_{p+l} < Y_{q+l}$. If $l < \min\{|X|-p, |Y|-q\}$, then the result follows immediately.

If $|X| - p \leq |Y| - q$ and $X_p\cdots X_{|X|-1} = Y_q\dots Y_{q+|X|-p-1}$, then $Y_{q+|X|-p} = R$, because otherwise we would have $Y_{q+|X|-p} = L$ and $Y_{q+|X|-p}\dots Y_{|Y|-1}Y^\infty > X^\infty$, and this violates admissibility of (X,Y). So $Y_{q+|X|-p} = R$ and, from Proposition 2 and Lemma 1,

$$(X,Y) * Z \leq Y^\infty \leq Y_{q+|X|-p}\dots Y_{|Y|-1}(X,Y) * Z', \tag{2}$$

and the result follows.

If $|X| - p \geq |Y| - q$ and $X_p\cdots X_{p+|Y|-q-1} = Y_q\cdots Y_{|Y|-1}$, then $X_{p+|Y|-q} = L$, because otherwise we would have $X_{p+|Y|-q} = R$ and $X_{p+|Y|-q}\cdots < Y^\infty$, which contradicts admissibility of (X,Y). So $X_{p+|Y|-q} = L$ and

$$X_{p+|Y|-q}\cdots X_{|X|-1}(X,Y) * Z \leq X^\infty \leq (X,Y) * Z' \tag{3}$$

and the result follows. ∎

Remark 3 *From the remark after Lemma 1, we can only have equalities in inequations 2 and 3 of the case specifically studied and analogous in the other cases of the proof if $s^m(X^\infty) = Y^\infty$ for some $0 < m < |X|$, so in the previous proposition, we can never have equality except if this happens.*

Proposition 4 *Let f be a (n,m)-renormalizable Lorenz map with $R_{(n,m)}(f) = g$, renormalization interval $[y_L, y_R]$ and kneading invariant $K_f = (X,Y) * (U,T)$, with $|X| = n$, $|Y| = m$ and $K_g = (U,T)$. Then*

$$i_f([y_L, y_R]) = \{(X,Y) * Z \text{ such that } Z \text{ is } g-admissible\}.$$

Proof

Note that $i_f(y_L) = (X_0\dots X_{|X|-1})^\infty = (X,Y) * L^\infty$ and $i_f(y_R) = (Y_0\dots Y_{|Y|-1})^\infty = (X,Y) * R^\infty$. Now, consider $x \in [y_L, y_R]$, if $x < 0$ then $(X,Y) * L^\infty \leq i_f(x) \leq$

$(X, Y) * U$, so the first $|X|$ symbols of $i_f(x)$ are equal to $X_0 \ldots X_{|X|-1}$. Analogously, if $y > 0$ then the first $|Y|$ symbols of $i_f(x)$ are equal to $Y_0 \ldots Y_{|Y|-1}$. Since f^n applies $[y_L, 0[$ in to $[y_L, y_R]$ and f^m applies $]0, y_R]$ in to $[y_L, y_R]$, we can repeat the previous argument to conclude that $i_f(x) = (X, Y) * Z$. The fact that Z is g-admissible and the reciprocal inclusion, follows immediately from Proposition 2.

∎

3 Lorenz knots and links

Let $n > 0$ be an integer. We denote by B_n the braid group on n strings given by the following presentation:

$$B_n = \left\langle \ \sigma_1, \sigma_2, \ldots, \sigma_{n-1} \ \middle| \ \begin{array}{ll} \sigma_i \sigma_j = \sigma_j \sigma_i & (|i - j| \geq 2) \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} & (i = 1, \ldots, n-2) \end{array} \ \right\rangle.$$

Where σ_i denotes a crossing between the strings occupying positions i and $i + 1$, such that the string in position i crosses (in the up to down direction) over the other, analogously σ_i^{-1}, the algebraic inverse of σ_i, denotes the crossing between the same strings, but in the negative sense, i.e., the string in position i crosses under the other. A *positive braid* is a braid with only positive crossings. A simple braid is a positive braid such that each two strings cross each other at most once. So there is a canonical bijection between the permutation group Σ_n and the set S_n, of simple braids with n strings, which associates to each permutation π, the braid b_π, where each point i is connected by a straight line to $\pi(i)$, keeping all the crossings positive.

Let X be a periodic sequence with least period k and let $\varphi \in \Sigma_k$ be the permutation that associates to each i, the position occupied by $s^i(X)$ in the lexicographic ordering of the k-tuple $(s(X), \ldots s^k(X))$ $(s^k(X) = X)$. Define $\pi \in \Sigma_k$ to be the permutation given by $\pi(\varphi(i)) = \varphi(i \mod k + 1)$, i.e., $\pi(i) = \varphi(\varphi^{-1}(i) + 1)$. We associate to π the corresponding simple braid $b_\pi \in B_k$ and call it the *Lorenz braid* associated to X. Since X is periodic, this braid represents a knot, and we call it the *Lorenz knot* associated to X.

Example: Let $X = (LRRLR)^\infty$. Hence we have $s^5(X) = X$, $s(X) = (RRLRL)^\infty$, $s^2(X) = (RLRLR)^\infty$, $s^3(X) = (LRLRR)^\infty$ and $s^4(X) = (RLRRL)^\infty$. Now after lexicographic reordering the $s^i(X)$ we obtain $s^3(X) < s^5(X) < s^2(X) < s^4(X) < s(X)$ and $\varphi = (1, 5, 2, 3)$ written as a disjoint cycle. Finally we obtain $\pi = (1, 4, 2, 5, 3)$ and $b_\pi = \sigma_2 \sigma_1 \sigma_3 \sigma_2 \sigma_4 \sigma_3$

We can also generalize the previous algorithm to be used in the case of a p-tuple of symbolic periodic sequences $(X^1, \ldots, X^p)$ with periods $(k_1, \ldots, k_p)$. In this case we proceed exactly as before with each one of the X^j. The permutation $\varphi \in \Sigma_{k_1 + \cdots + k_p}$ is the permutation that describes the lexicographic ordering of the $(k_1 + \cdots k_p)$-tuple $(s(X^1), \ldots, s^{k_1}(X^1), \ldots, s(X^p) \ldots, s^{k_p}(X^p))$ and $\pi \in \Sigma_{k_1 + \cdots + k_p}$ is defined by $\pi(\varphi(i)) = \varphi(i + 1)$ if there is no q such that $i = k_1 + \cdots + k_q$ and $\pi(\varphi(i)) = \varphi(k_1 + \cdots + k_{q-1} + 1)$ if $i = k_1 + \cdots k_q$, assuming $k_0 = 0$.

Remark 4 *What we are doing here is simply to mark in two parallel lines, $k_1 + \ldots + k_p$ points, corresponding in a ordered way, to the sequences $s^{i_j}(X^j)$, $j = 1, \ldots, p$, $i_j = 1, \ldots, k_j$ and connect by straight lines the points corresponding to $s^{i_j}(X^j)$ with the points corresponding to $s^{i_j+1}(X^j)$, keeping the crossings positive.*
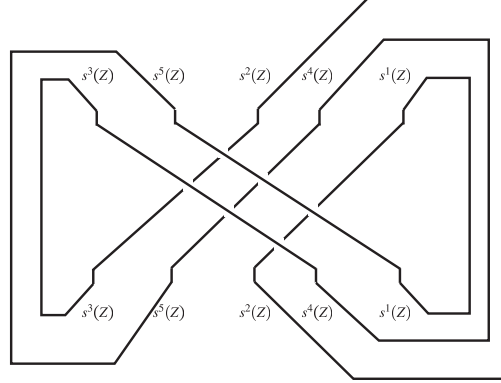
Figure 3: The Lorenz knot associated to $X = (LRRLR)^{\infty}$

4   The renormalization subtemplate

A *template* is a compact branched two-manifold with boundary and a smooth expansive semiflow built locally from two types of charts: joining and splitting (see Figure 1). Each chart carries a semiflow, endowing the template with an expanding semiflow, and the gluing maps between charts must reflect the semiflow and act linearly on the edges.

Following [4], we can take a semigroup structure on braided templates. The generators of the braided template semigroup are:

1. $\sigma_i^{\pm}$, a positive (resp.negative) crossing between the strips occupying the i-th and $(i+1)$-th positions;

2. $\tau_i^{\pm}$, a half twist in the strip occupying the i-th position, in the positive (resp. negative) sense;

3. $\beta_i^{\pm}$, a branch line chart with the i-th and $(i+1)$-th strips incoming , 2 outgoing strips and either a positive (β_i) or negative (β_i^-) crossing at the branch line.
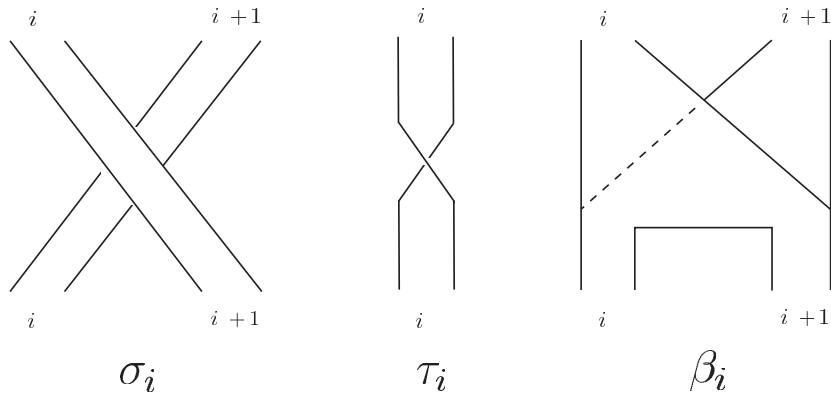

Figure 4: Generators of the braided template semigroup

Given any pair of finite admissible sequences (X, Y), we define the *tail's length* $m(X, Y)$ as

$$m(X, Y) = \min\{i \geq 0 : X_{|X|-1-i} \neq Y_{|Y|-1-i}\}$$

For a finite sequence S, let $n_L(S) = \#\{S_i : 0 \leq i < |S| \text{ and } S_i = L\}$, $n_R(S) = \#\{S_i : 0 \leq i < |S| \text{ and } S_i = R\}$.

Now, to any finite admissible pair (X, Y), we associate a subtemplate $R(X, Y)$, the *renormalization subtemplate* associated to (X, Y), on the following way: Consider the Lorenz braid associated to (X, Y), whose word is $\sigma_{p_1} \cdots \sigma_{p_k}$. Consider the relative position $j = \varphi(|X| - m(X, Y))$, of $s^{|X|-m(X,Y)}(X)$.

If $s^n(X^\infty) = Y^\infty$ for some $n < |X|$, then the sequences X and Y generate the same Lorenz knot and we consider the Lorenz link associated to (X, Y) with only one component. In this case $R(X, Y)$ is the template with $|X|$ strips and word $\sigma_{p_1} \cdots \sigma_{p_k} \beta_j^\pm$, where the signal $+$ in β_j is taken if $X_{|X|-m(X,Y)-1} = L$ and the signal $-$ is taken otherwise.

If $s^n(X^\infty) \neq Y^\infty$ for all $n < |X|$ then $R(X, Y)$ is the template with $|X| + |Y|$ strips and word $\sigma_{p_1} \cdots \sigma_{p_k} \beta_j^\pm$, where the signal $+$ in β_j is taken if $X_{|X|-m(X,Y)-1} = L$ and the signal $-$ is taken otherwise.
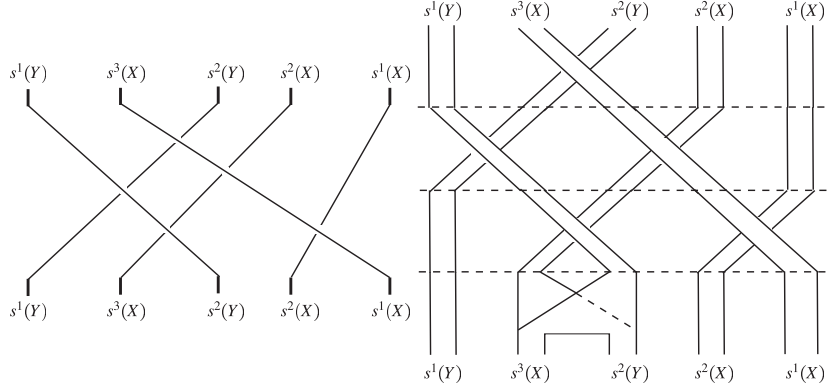

Figure 5: The renormalization template associated to $(X, Y) = ((LRR)^\infty, (RL)^\infty$

Remark 5 *What we are doing is simply to substitute each string of the braid associated to (X, Y) by a strip and add $\beta_j^\pm$ according if $X_{|X|-m(X,Y)-1} = L$ or $X_{|X|-m(X,Y)-1} = R$, respectively, see Figure 5*

The next theorem is naturally motivated from Proposition 4.

Theorem 3 *Let (X, Y) be one admissible pair of finite sequences and $(Z^1, \ldots, Z^n)$ be a n-tuple of sequences whose associated Lorenz link haves braid word $\sigma_{p_1} \cdots \sigma_{p_k}$, then the Lorenz link associated to $((X, Y) * Z^1, \ldots, (X, Y) * Z^n)$ is the Lorenz link contained in $R(X, Y)$ with:*

1. *$|Z^1| + \cdots + |Z^n|$ strings in each strip if $s^i(X^\infty) = Y^\infty$ for some $i < |X|$.*

2. *$n_L(Z^1) + \cdots + n_L(Z^n)$ strings in each strip associated to X and $n_R(Z^1) + \cdots + n_R(Z^n)$ strings in each strip associated to Y if $s^i(X^\infty) \neq Y^\infty$ for all $i < |X|$.*

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In both cases, the braid word of the restriction to the branch line chart β_j (respectively β_j^-) is $\sigma_{q+p_1} \cdots \sigma_{q+p_k}$ (respectively $\sigma_{q+p_1}^{-1} \cdots \sigma_{q+p_k}^{-1}$), where $q+1$ is the index of the left-most string getting in β_j.
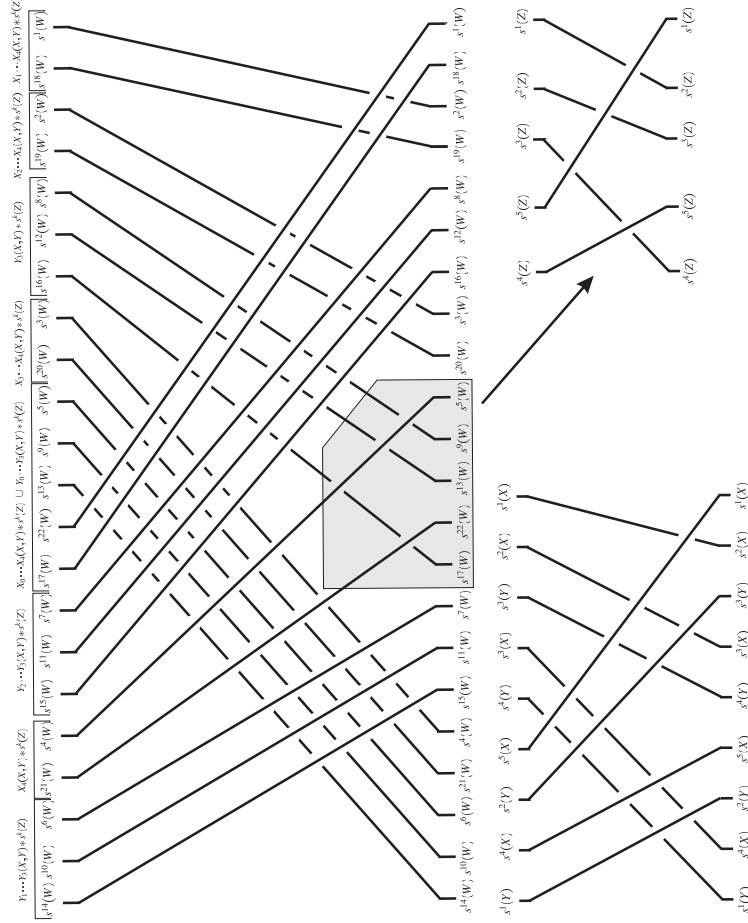

Figure 6: Pictoric ilustration of the Theorem: the Lorenz braid associated to $W = (X,Y) * Z = ((LRRRL)^\infty, (RLLR)^\infty) * (LRRRL)^\infty$ on the top, the Lorenz braids associated to (X,Y) and Z on the bottom.

Proof

We will only consider the case $s^i(X^\infty) \neq Y^\infty$ for all $i < |X|$, since the proof of the other case is completely analogous.

Without lost of generality, we can consider $n = 1$, i.e. the Lorenz knot associated to $(X, Y) * Z$.

Consider the permutations $\varphi_{(X,Y)}$, φ_Z and $\varphi_{(X,Y)*Z}$ associated with the lexicographic ordering of the sequences $(s(X), \ldots, s^{|X|}(X), s(Y), \ldots, s^{|Y|}(Y))$, $(s(Z), \ldots, s^{|Z|}(Z))$ and $(s((X,Y)*Z), \ldots, s^{|(X,Y)*Z|}((X,Y)*Z))$, respectively. Analogously consider $\pi_{(X,Y)}$, π_Z and $\pi_{(X,Y)*Z}$, the permutation induced by the shift map over the respective lexicographically ordered sequences (see Section 3).

Let

$$W^k = \begin{cases} X \text{ if } Z_{\varphi_Z^{-1}(k)} = L \\ Y \text{ if } Z_{\varphi_Z^{-1}(k)} = R \end{cases}$$

For each $1 \leq k \leq |Z|$ and $0 \leq p \leq |W^k| - 1$, define

$$\begin{aligned} \Phi(p,k) &= \varphi_{(X,Y)*Z}\big(s^p((X,Y)*s^{\varphi_Z^{-1}(k)}(Z))\big) \\ &= \varphi_{(X,Y)*Z}\big(W_p^k \ldots W_{|W^k|-1}^k(X,Y)*s^{\varphi_Z^{-1}(k)+1}(Z)\big). \end{aligned}$$

From Propositions 2 and 3, for each $1 \leq i \leq n_L(Z) - 1$ and $0 \leq p \leq |X| - m(X,Y) - 1$, we have that

$$\Phi(p, i+1) = \Phi(p, i) + 1, \tag{4}$$

and, analogously, for each $n_L(Z) + 1 \leq j \leq |Z| - 1$ and $0 \leq q \leq |Y| - m(X,Y) - 1$

$$\Phi(q, j+1) = \Phi(q, j) + 1, \tag{5}$$

This means that, in the lexicographic ordering of $s^i((X,Y)*Z)$, the sequences $W_p \cdots W_{|W|-1}(X,Y)*s^k(Z)$ are all disposed together, constituting a set of $n_L(Z)$ sequences if $W = X$ and of $n_R(Z)$ sequences if $W = Y$, ordered by the lexicographic ordering of $s^k(Z)$.

Moreover,

$$s(W_p \cdots W_{|W|-1}(X,Y)*s^{\varphi_Z^{-1}(k)+1}(Z)) = W_{p+1} \cdots W_{|W|-1}(X,Y)*s^{\varphi_Z^{-1}(k)+1}(Z),$$

this means that

$$\pi_{(X,Y)*Z}(\Phi(p,k)) = \Phi(p+1, k),$$

so there are exactly $n_L(Z)$ (resp. $n_R(Z)$) parallel strings from the set

$$\{\Phi(p,k), k = 1, \cdots, n_L(Z)\} \text{ to } \{\Phi(p+1,k), k = 1, \cdots, n_L(Z)\}$$

(resp. from

$$\{\Phi(p,k), k = n_L(Z)+1, \cdots, |Z|\} \text{ to } \{\Phi(p+1,k), k = n_L(Z)+1, \cdots, |Z|\}).$$

Let us now consider the case $p \geq |W^k| - m(X,Y)$:
Since $\varphi_Z^{-1}(\pi_Z^{-1}(k)) = \varphi_Z^{-1}(k) - 1$, we have that

$$\pi_Z^{-1}(k) = \varphi_Z(\varphi_Z^{-1}(k) - 1),$$

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so, if $1 \le l \le m(X,Y)$ then

$$\Phi(|W^{\pi_Z^{-1}(i)}|-l, \pi_Z^{-1}(i)) = \varphi_{(X,Y)*Z}(W_{|W^{\pi_Z^{-1}(i)}|-l}^{\pi_Z^{-1}(i)} \cdots W_{|W^{\pi_Z^{-1}(i)}|-1}^{\pi_Z^{-1}(i)}(X,Y)*s^{\varphi_Z^{-1}(i)}(Z)).$$

From Propositions 2 and 3, for each $1 \le i \le |Z|-1$ and $1 \le l \le m(X,Y)$, we have that

$$\Phi(|W^{\pi_Z^{-1}(i+1)}| - l, \pi_Z^{-1}(i+1)) = \Phi(|W^{\pi_Z^{-1}(i)}| - l, \pi_Z^{-1}(i)) + 1 \tag{6}$$

moreover, if $l > 1$ then

$$\pi_{(X,Y)*Z}(\Phi(|W^i| - l, i) = \Phi(|W^i| - l + 1, i).$$

So, in the lexicographic ordering of $s^i((X,Y)*Z)$, the sequences $W_{|W|-l} \cdots W_{|W|-1}(X,Y)*s^k(Z)$ (with $W = W^{\varphi_Z(k-1)}$) are all disposed together, constituting a set of $|Z|$ sequences ordered according with $\pi_Z^{-1}(k)$ and there are exactly $|Z|$ parallel strings from the set

$$\{\Phi(|W^k| - l, k), k = 1, \ldots, |Z|\} \text{ to } \{\Phi(|W^k| - l + 1, k), k = 1, \ldots, |Z|\}.$$

From Proposition 3, the strings in the Lorenz link associated to (X,Y), corresponding to $s^{|X|-l}(X) \mapsto s^{|X|-l+1}(X)$ and $s^{|Y|-l}(Y) \mapsto s^{|Y|-l+1}(Y)$ are parallel without any other string between them. So we only have to divide each set of $|Z|$ strings from $\{\Phi(|W^k| - l, k), k = 1, \cdots, |Z|\}$ to $\{\Phi(|W^k| - l + 1, k), k = 1, \cdots, |Z|\}$ in two subsets, the one on the left with $n_L(X)$ strings contained in the strip associated to $s^{|X|-l}(X) \mapsto s^{|X|-l+1}(X)$ and the one on the right with $n_R(Z)$ strings contained in the strip associated to $s^{|Y|-l}(Y) \mapsto s^{|Y|-l+1}(Y)$.

From Proposition 3, $s^p(W^k) < s^q(W^{k'})$ implies that $\Phi(p,k) < \Phi(q,k')$, so the sets $\{\Phi(p,k), k\}_p$ are ordered according with $\varphi_{(X,Y)}$, this implies that the transitions between this sets are done according with the respective transitions in the Lorenz braid associated to (X,Y). This finishes the proof of the first part of the theorem.

To see what happens in the branch line chart we must look to the transition to the tail, i.e. $l = m(X,Y) + 1$:

If $X_{|X|-m(X,Y)-1} = L$ then $Y_{|Y|-m(X,Y)-1} = R$ and $\Phi(|X| - m(X,Y) - 1, k) < \Phi(|Y| - m(X,Y) - 1, k')$ for all $1 \le k \le n_L(Z)$ and $n_L(Z) < k' \le |Z|$.

On the other hand,

$$s(W_{|W^k|-m(X,Y)-1}^k \cdots W_{|W^k|-1}^k(X,Y) * s^{\varphi_Z^{-1}(k)+1}(Z)) =$$

$$\begin{cases} W_{|W^k|-m(X,Y)}^k \cdots W_{|W^k|-1}^k(X,Y) * s^{\varphi_Z^{-1}(k)+1}(Z) \text{ if } m(X,Y) > 0 \\ (X,Y) * s^{\varphi_Z^{-1}(k)+1}(Z) \text{ if } m(X,Y) = 0 \end{cases} \tag{7}$$

and, while, from 4 and 5, the elements $W_{|W^k|-m(X,Y)-1}^k \cdots W_{|W^k|-1}^k(X,Y)*s^{\varphi_Z^{-1}(k)+1}(Z)$ are ordered according with k, from 6, their shift images are ordered according with $\pi_Z^{-1}(k)$, this means that the permutation given by the strings connecting the two sets is exactly π_Z.

If $X_{|X|-m(X,Y)-1} = R$ then $Y_{|Y|-m(X,Y)-1} = L$ and

$$\pi_{(X,Y)}(\varphi_{(X,Y)}(s^{|X|-m(X,Y)-1}(X)) < \pi_{(X,Y)}(\varphi_{(X,Y)}(s^{|Y|-m(X,Y)-1}(Y)),$$

this generates the crossing σ_j in the braid associated to (X,Y). Regarding to the braid associated to $(X,Y) * Z$, we have that

$$\Phi(|Y| - m(X,Y) - 1, k') < \Phi(|X| - m(X,Y) - 1, k)$$

for all $1 \leq k \leq n_L(Z)$ and $n_L(Z) < k' \leq |Z|$. Now, considering

$$\xi(k) = \left\{ \begin{array}{l} n_L(Z) + k \text{ if } 1 \leq k \leq n_R(Z) \\ k - n_R(Z) \text{ if } n_R(Z) < k \leq |Z|, \end{array} \right.$$

while the elements $W^k_{|W^k|-m(X,Y)-1} \cdots W^k_{|W^k|-1}(X,Y) * s^{\varphi_Z^{-1}(k)+1}(Z)$ are ordered according with $\xi(k)$, their shift images are ordered according with $\pi_Z^{-1}(k)$, this means that the permutation given by the strings connecting the two sets, after crossing the X-strip with the Y-strip, is exactly π_Z so, from Remark 6 below, the braid restricted to the branch line chart is exactly $\sigma_{p_1}^{-1} \cdots \sigma_{p_k}^{-1}$. $\blacksquare$

Remark 6 *The Reidemeister moves induce relations that are verified on the braid group (resp. braided template semigroup). In the braided template semigroup, one of these relations is $\sigma_i \beta_i^- = \beta_i$. This corresponds to make a Reidemeister Type II move with the $i-th$ and the $i+1-th$ strips, inverting the crossing on the β_i line chart (changing the sign), see Figure 7.*

Given a simple braid b (or the corresponding permutation) in a branch line chart β_i, it can be decomposed as the product of two braids. This decomposition $b = sb'$ is such that s is a simple braid in the chart σ_i and b' is a mirrored simple braid (obtained from a simple braid changing all the crossings from positive to negative) in β_i^-.
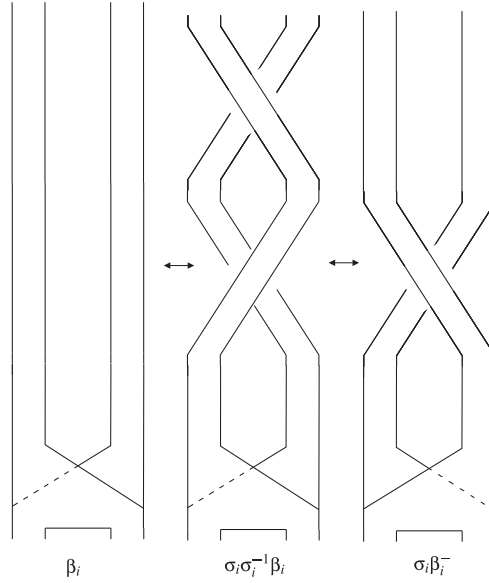

Figure 7: Reidemeister Type II move and the relation $\sigma_i \beta_i^- = \beta_i$.

5   Invariants

We will start this section, introducing some terminology, following [2].

Let β be a Lorenz braid:

1. The *string index* is the number n of strings in β. It is the sum of the word lengths.

2. The *braid index* of a knot is the minimum string index among all closed braid representatives of that knot.

3. The *crossing number* c is the number of double points in the projected image of the Lorenz braid β.

4. The *linking number* $l(X, Y)$ is the number of crossings between one string from the knot associated to X and one string from the knot associated to Y.

5. The *genus* g of a link L is the genus of M, where M is an orientable surface of minimal genus spanned by L.

Remark 7 *Through all over this section we will only consider admissible pairs of finite sequences (X, Y), such that $s^n(X^\infty) \neq Y^\infty$ for all $n < |X|$. All the results respective to the case $s^n(X^\infty) = Y^\infty$, follows analogously.*

Lemma 2 *Let (X, Y) be a finite admissible pair and S a finite sequence. Then the crossing number $c((X, Y) * S)$ of the Lorenz braid associated to $(X, Y) * S$, is given by*

$$c((X, Y) * S) = c(X)n_L(S)^2 + c(Y)n_R(S)^2 + l(X, Y)n_L(S)n_R(S) \pm c(S).$$

where we take the signal $+$ in $c(S)$ if $X_{|X|-m(X,Y)-1} = L$ and the signal $-$ otherwise.

    Proof: There are four contributions to the computation of $c((X, Y) * S)$. Two of them come from $c(X)$ and $c(Y)$, the third one from $l(X, Y)$ and the fourth from $c(S)$. So $c(X)$ will be counted $n_L(S)^2$ times and $c(Y)$ will be counted $n_R(S)^2$ times, this corresponds to the substitution of one crossing on X (resp. Y) by the $n_L(S)^2$ (resp. $n_R(S)^2$) crossings arising from inflating each X-string (resp. each Y-string) with $n_L(S)$ (resp. $n_R(S)$) strings. Similarly for each crossing counted in $l(X, Y)$ we obtain $n_L(S) \times n_R(S)$ crossings. Finally we must count the crossings in β_j, and, from the Main Theorem and Remark 6, this means to add or subtract $c(S)$ according to if $X_{|X|-m(X,Y)-1} = L$ or $X_{|X|-m(X,Y)-1} = R$. ∎

Lemma 3 *Let (X, Y) and (S, W) be finite admissible pairs. Then the linking number $l((X, Y) * (S, W))$ of the Lorenz braid associated to $(X, Y) * (S, W)$, is given by*

$$l((X, Y) * (S, W)) = l((X, Y) * S, (X, Y) * W) =$$
$$2c(X)n_L(S)n_L(W) + 2c(Y)n_R(S)n_R(W)+$$
$$l(X, Y)(n_L(W)n_R(S) + n_R(W)n_L(S)) \pm l(S, W)$$

    where we take the signal $+$ in $l(S, W)$ if $X_{|X|-m(X,Y)-1} = L$ and the signal $-$ otherwise.

    Proof: The proof is analogous to the proof of the previous lemma , except that in this case we have to count the crossings in the link $(X, Y) * (S, W) = ((X, Y)*S, (X, Y)*W)$ between a string from $(X, Y)*S$ and a string from $(X, Y)*W$. ∎

Let (S, W) be a finite admissible pair. We denote

$$A_{33} = \begin{bmatrix} n_L(S)^2 & n_L(W)^2 & 2n_L(S)n_L(W) \\ n_R(S)^2 & n_R(W)^2 & 2n_R(S)n_R(W) \\ n_L(S)n_R(S) & n_L(W)n_R(W) & n_L(W)n_R(S) + n_L(S)n_R(W) \end{bmatrix}$$

and

$$B_{13} = \begin{bmatrix} (n_L(S) + n_L(W))^2 \\ (n_R(S) + n_R(W))^2 \\ (n_L(S) + n_L(W))(n_R(S) + n_R(W)) \end{bmatrix}$$

Lemma 4 *Let (X, Y) and (S, W) be finite admissible pairs. Then*

$$c((X, Y) * (S, W)) = \begin{bmatrix} c(X) & c(Y) & l(X, Y) \end{bmatrix} B_{13} \pm c((S, W))$$

where we take the signal $+$ in $c(S, W)$ if $X_{|X|-m(X,Y)-1} = L$ and the signal $-$ otherwise.

Proof: The result follows from the previous two lemmas, observing that $c((X, Y) * (S, W)) = c((X, Y) * S) + c((X, Y) * W) + l((X, Y) * (S, W))$. ∎

Remark 8 *We are concerned with the behavior of the invariants mentioned above, through sequences of Lorenz braids and knots associated to kneading invariants of type $(A(n), B(n)) = (X, Y) * (S, W)^n = (X, Y) * (S, W)^{n-1} * (S, W)$. Because of the phenomenon described in Remark 6 this invariants may depend on the symbols $A(n-1)_{|A(n-1)|-m(A(n-1),B(n-1))-1}$, so in the following we must consider four different cases:*

1. *If $X_{|X|-m(X,Y)-1} = S_{|S|-m(S,W)-1} = L$, then $A(n)_{|A(n)|-m(A(n),B(n))-1} = L$ for all n.*

2. *If $X_{|X|-m(X,Y)-1} = L$ and $S_{|S|-m(S,W)-1} = R$, then*

$$A(n)_{|A(n)|-m(A(n),B(n))-1} = \begin{cases} L & \text{if } n \text{ is even} \\ R & \text{if } n \text{ is odd} \end{cases}$$

   .

3. *If $X_{|X|-m(X,Y)-1} = R$ and $S_{|S|-m(S,W)-1} = L$, then $A(n)_{|A(n)|-m(A(n),B(n))-1} = R$ for all n.*

4. *If $X_{|X|-m(X,Y)-1} = S_{|S|-m(S,W)-1} = R$, then*

$$A(n)_{|A(n)|-m(A(n),B(n))-1} = \begin{cases} L & \text{if } n \text{ is odd} \\ R & \text{if } n \text{ is even} \end{cases}$$

   .

Lemma 5 *Let (X, Y) and (S, W) be finite admissible pairs, then, for $n \geq 2$ we have that*

$$c((X, Y) * (S, W)^n) =$$
$$\left(\begin{bmatrix} c(X) & c(Y) & l(X, Y) \end{bmatrix} A_{33}^{n-1} + \begin{bmatrix} c(S) & c(W) & l(S, W) \end{bmatrix} \sum_{i=0}^{n-2} a_i A_{33}^i \right) B_{13}$$
$$+ \alpha c(S, W)$$

where, considering the four cases of Remark 8, we have: in Case 1 $a_i = 1 = \alpha$ for all i; in Case 2 $a_i = (-1)^{i+n}$ and $\alpha = (-1)^{n+1}$; in Case 3 $a_i = -1 = \alpha$ for all i; in Case 4 $a_i = (-1)^{i+n+1}$ and $\alpha = (-1)^n$.

Proof: We will construct the formula recursively.

For $n = 2$, we want to compute $c((X,Y)*(S,W)^2) = c((X,Y)*(S,W)*(S,W))$. Using first Lemma 4 and then Lemmas 2 and 3, always considering Remark 8, we have

$$c((X,Y)*(S,W)*(S,W)) = c(((X,Y)*S,(X,Y)*W)*(S,W)) =$$

$$\begin{bmatrix} c((X,Y)*S) & c((X,Y)*W) & l((X,Y)*(S,W)) \end{bmatrix} B_{13} \pm_1 c(S,W) =$$

$$\left(\begin{bmatrix} c(X) & c(Y) & l(X,Y) \end{bmatrix} A_{33} \pm_0 \begin{bmatrix} c(S) & c(W) & l(S,W) \end{bmatrix}\right) B_{13} \pm_1 c(S,W)$$

where $\pm_0$ depends on $X_{|X|-m(X,Y)-1}$ and $\pm_1$ depends on $A(1)_{|A(1)|-m(A(1),B(1))-1}$.

For $n = 3$, we want to compute $c((X,Y)*(S,W)^3) = c((X,Y)*(S,W)^2*(S,W))$. As in case $n = 2$ we obtain

$$c((X,Y)*(S,W)^2*(S,W)) =$$
$$c(((X,Y)*(S,W)*S,(X,Y)*(S,W)*W)*(S,W)) =$$
$$\begin{bmatrix} c((X,Y)*(S,W)*S) & c((X,Y)*(S,W)*W) & l((X,Y)*(S,W)^2) \end{bmatrix} B_{13}$$
$$\pm_2 c(S,W) =$$
$$\begin{bmatrix} c((X,Y)*S) & c((X,Y)*W) & l((X,Y)*(S,W)) \end{bmatrix} A_{33}B_{13}$$
$$\pm_1 \begin{bmatrix} c(S) & c(W) & l(S,W) \end{bmatrix} B_{13} \pm_2 c(S,W) =$$
$$\left(\begin{bmatrix} c(X) & c(Y) & l(X,Y) \end{bmatrix} A_{33} \pm_0 \begin{bmatrix} c(S) & c(W) & l(S,W) \end{bmatrix}\right) A_{33}B_{13}$$
$$\pm_1 \begin{bmatrix} c(S) & c(W) & l(S,W) \end{bmatrix} B_{13} \pm_2 c(S,W) =$$
$$\begin{bmatrix} c(X) & c(Y) & l(X,Y) \end{bmatrix} A_{33}^2 B_{13} \pm_0 \begin{bmatrix} c(S) & c(W) & l(S,W) \end{bmatrix} A_{33}B_{13}$$
$$\pm_1 \begin{bmatrix} c(S) & c(W) & l(S,W) \end{bmatrix} B_{13} \pm_2 c(S,W)$$

where, considering $(A(0),B(0)) = (X,Y)$, $\pm_i$ depends on $A(i)_{|A(i)|-m(A(i),B(i))-1}$.

We now obtain the formula recursively. ∎

Lemma 6 *Let $n \in \mathbb{N}$ and (X,Y) and (S,W) be finite admissible pairs. Then the string index of the Lorenz braid associated to $(X,Y)*(S,W)^n$ is*

$$|(X,Y)*(S,W)^n| = [|X||Y|] \begin{bmatrix} n_L(S) & n_L(W) \\ n_R(S) & n_R(W) \end{bmatrix}^n \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Proof: From the definition of $*$-product, we have that

$$|(X,Y)*(S,W)| = |((X,Y)*S,(X,Y)*W)| = |(X,Y)*S| + |(X,Y)*W| =$$

$$(n_L(S) + n_L(W))|X| + (n_R(S) + n_R(W))|Y| =$$

$$\begin{bmatrix} |X| & |Y| \end{bmatrix} \begin{bmatrix} n_L(S) & n_L(W) \\ n_R(S) & n_R(W) \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Now, by induction,

$$|(X,Y)*(S,W)^{n+1}| = |((X,Y)*S,(X,Y)*W)*(S,W)^n| =$$

$$\begin{bmatrix} |(X,Y)*S| & |(X,Y)*W| \end{bmatrix} \begin{bmatrix} n_L(S) & n_L(W) \\ n_R(S) & n_R(W) \end{bmatrix}^n \begin{bmatrix} 1 \\ 1 \end{bmatrix} =$$

$$\begin{bmatrix} |X| & |Y| \end{bmatrix} \begin{bmatrix} n_L(S) & n_L(W) \\ n_R(S) & n_R(W) \end{bmatrix}^{n+1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

■

5.1   Braid index

The *trip number*, t, of a finite sequence X, is the number of syllables in X, a syllable being a maximal subword of X, of the form $L^a R^b$.

Birmann and Williams conjectured in [2] that, for the case of a Lorenz knot τ, $b(\tau) = t(\tau)$, where $t(\tau)$ is the trip number of the finite sequence associated to τ. In [12], following a result obtained by Franks and Williams in [8], Waddington observed that this conjecture is true. So our computations will be done about t.

Proposition 5 (Trip number and Braid index) *Let (X, Y) be a finite admissible pair, and S be a finite sequence, then we have:*

1. *If $X_{|X|-1} = Y_{|Y|-1}$, then*

$$t((X, Y) * S) = n_L(S)t(X) + n_R(S)t(Y).$$

2. *If $X_{|X|-1} \neq Y_{|Y|-1}$, then*

$$t((X, Y) * S) = n_L(S)t(X) + n_R(S)t(Y) \pm t(S),$$

where we take the signal $+$ in $t(S)$ if $X_{|X|-1} = L$ and signal $-$ otherwise.

Proof: The trip number is equal to the number of strings that travel fom the L-side to the R-side (or, equivalently, from the R-side to the L-side). In Case 1, since the branch line chart β_j is located completely in the L-side if $X_{|X|-m(X,Y)} = L$ or in the R-side if $X_{|X|-m(X,Y)} = R$, the only contributions to the trip number come from the strips relative to X and relative to Y that travel from the L-side to the R-side. Since there are exactly $n_L(S)$ strings in each X-strip and $n_R(S)$ in each Y-strip, we get the result. In the second case we have $j = n_L(X) + n_L(Y)$, this means that β_j has one incoming strip from the L-side, other from the R-side and the outgoing strips are also one in the L-side and other in the R-side, so, from the Main theorem and Remark 6, the β_j chart will contribute with $\pm t(S)$ strings from the L-side to the R-side. ■

Proposition 6 *Let (X, Y) and (S, W) be finite admissible pairs. Then, for each $n \in \mathbb{N}$, we have:*

1. *If $X_{|X|-1} = Y_{|Y|-1}$, then*

$$\left[\begin{array}{c} t((X, Y) * (S, W)^{n-1} * S) \\ t((X, Y) * (S, W)^{n-1} * W) \end{array} \right] = \left[\begin{array}{cc} n_L(S) & n_R(S) \\ n_L(W) & n_R(W) \end{array} \right]^n \left[\begin{array}{c} t(X) \\ t(Y) \end{array} \right].$$

2. *If $X_{|X|-1} \neq Y_{|Y|-1}$ and $S_{|S|-1} \neq W_{|W|-1}$, then*

$$\left[\begin{array}{c} t((X, Y) * (S, W)^{n-1} * S) \\ t((X, Y) * (S, W)^{n-1} * W) \end{array} \right] = \left[\begin{array}{cc} n_L(S) & n_R(S) \\ n_L(W) & n_R(W) \end{array} \right]^n \left[\begin{array}{c} t(X) \\ t(Y) \end{array} \right]$$

$$+ \sum_{i=0}^{n-1} a_i \left[\begin{array}{cc} n_L(S) & n_R(S) \\ n_L(W) & n_R(W) \end{array} \right]^i \left[\begin{array}{c} t(S) \\ t(W) \end{array} \right]$$

where, considering the cases from Remark 8,: in Case 1 $a_i = 1$ for all i; in Case 2 $a_i = (-1)^{i+n+1}$; in Case 3 $a_i = -1$ for all i; in Case 4 $a_i = (-1)^{i+n}$.

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3. If $X_{|X|-1} \neq Y_{|Y|-1}$ and $S_{|S|-1} = W_{|W|-1}$, then

$$\begin{bmatrix} t((X,Y)*(S,W)^{n-1}*S) \\ t((X,Y)*(S,W)^{n-1}*W) \end{bmatrix} = \begin{bmatrix} n_L(S) & n_R(S) \\ n_L(W) & n_R(W) \end{bmatrix}^{n-1} \begin{bmatrix} t((X,Y)*S) \\ t((X,Y)*W) \end{bmatrix} =$$
$$\begin{bmatrix} n_L(S) & n_R(S) \\ n_L(W) & n_R(W) \end{bmatrix}^{n-1} \left(\begin{bmatrix} n_L(S) & n_R(S) \\ n_L(W) & n_R(W) \end{bmatrix} \begin{bmatrix} t(X) \\ t(X) \end{bmatrix} \pm \begin{bmatrix} t(S) \\ t(W) \end{bmatrix} \right)$$

where we take the signal $+$ in the last summand if $X_{|X|-1} = L$ and the signal $-$ otherwise.

Proof: This proof will be done by induction on n. The case $n = 1$ is just Proposition 5. Now to prove the induction step.

1. Suppose that the formula in case 1 is true for n. So we want to compute

$$\begin{bmatrix} t((X,Y)*(S,W)^n*S) \\ t((X,Y)*(S,W)^n*W) \end{bmatrix} =$$
$$= \begin{bmatrix} t(((X,Y)*(S,W)^{n-1}*S,(X,Y)*(S,W)^{n-1}*W)*S) \\ t(((X,Y)*(S,W)^{n-1}*S,(X,Y)*(S,W)^{n-1}*W)*W) \end{bmatrix}$$

Hence from Proposition 5:

$$\begin{bmatrix} t((X,Y)*(S,W)^n*S) \\ t((X,Y)*(S,W)^n*W) \end{bmatrix} =$$

$$= \begin{bmatrix} n_L(S)t(((X,Y)*(S,W)^{n-1}*S) + n_R(S)t((X,Y)*(S,W)^{n-1}*W) \\ n_L(W)t(((X,Y)*(S,W)^{n-1}*S) + n_R(W)t((X,Y)*(S,W)^{n-1}*W) \end{bmatrix} =$$

$$= \begin{bmatrix} n_L(S) & n_R(S) \\ n_L(W) & n_R(W) \end{bmatrix} \begin{bmatrix} t(((X,Y)*(S,W)^{n-1}*S) \\ t((X,Y)*(S,W)^{n-1}*W) \end{bmatrix}$$

and we can apply our hypothesis to the second factor.

2. To prove case 2 we will follow the same steps as in case 1 to obtain

$$\begin{bmatrix} t((X,Y)*(S,W)^n*S) \\ t((X,Y)*(S,W)^n*W) \end{bmatrix} =$$

$$\begin{bmatrix} n_L(S)t(((X,Y)*(S,W)^{n-1}*S) + n_R(S)t((X,Y)*(S,W)^{n-1}*W) \pm t(S) \\ n_L(W)t(((X,Y)*(S,W)^{n-1}*S) + n_R(W)t((X,Y)*(S,W)^{n-1}*W) \pm t(W) \end{bmatrix} =$$

$$\begin{bmatrix} n_L(S) & n_R(S) \\ n_L(W) & n_R(W) \end{bmatrix} \begin{bmatrix} t(((X,Y)*(S,W)^{n-1}*S) \\ t((X,Y)*(S,W)^{n-1}*W) \end{bmatrix} \pm \begin{bmatrix} t(S) \\ t(W) \end{bmatrix}$$

where the signal $\pm$ in the last summand depends on Remark 8. Once again we obtain the desired result by applying our hypothesis to the second factor of the first part of the sum .

3. Since $S_{|S|-1} = W_{|W|-1}$ then $A(1)_{|A(1)|-1} = B(1)_{|B(1)|-1}$, so, because $(X,Y)*(S,W)^n = ((X,Y)*(S,W))*(S,W)^{n-1}$ we can apply part 1 of this proposition to $((X,Y)*(S,W))*(S,W)^{n-1}$ and then part 2 of Proposition 5 to $(X,Y)*(S,W)$ ∎

5.2 Genus

From Theorem 1.1.18 of [4], given a link K and a braid representative b_K of the link, we have

$$g(K) = \frac{C - N - u}{2} + 1, \tag{8}$$

where C is the number of crossings in b_K, N the string index and u the number of link components. We want now to compute $g((X,Y) * S)$.

Proposition 7 (Genus for knots) *Let (X,Y) be a finite admissible pair and S be a finite sequence. Then the genus of the knot associated to $(X,Y) * S$ is given by:*

$$g((X,Y) * S) =$$

$$\frac{c(X)n_L(S)^2 + c(Y)n_R(S)^2 + l(X,Y)n_L(S)n_R(S) - n_L(S)|X| - n_R(S)|Y| + 1 \pm c(S)}{2},$$

where we take the signal $+$ in $c(S)$ if $X_{|X|-m(X,Y)-1} = L$ and the signal $-$ otherwise.

Proof: First notice that, because $(X,Y) * S$ is a knot we have $u = 1$. Now the number of strings in $(X,Y) * S$ is equal to $|(X,Y) * S)| = n_L(S)|X| + n_R(S)|Y|$. The value of $c((X,Y) * S)$ is given by Lemma 2. So

$$g((X,Y) * S) = \frac{c((X,Y) * S) - |(X,Y) * S)| + 1}{2} =$$

$$\frac{c(X)n_L(S)^2 + c(Y)n_R(S)^2 + l(X,Y)n_L(S)n_R(S) - n_L(S)|X| - n_R(S)|Y| + 1 \pm c(S)}{2}$$

∎

Proposition 8 (Genus for links) *Let (X,Y) and (S,W) be finite admissible pairs. Then the genus of the Lorenz link associated to $(X,Y) * (S,W)$ is given by:*

$$g((X,Y) * (S,W)) = \frac{c(X)(n_L(S)^2 + n_L(W)^2) + c(Y)(n_R(S)^2 + n_R(W)^2)}{2} +$$

$$\frac{l(X,Y)(n_L(S)n_R(S) + n_L(W)n_R(W)) \pm (c(W) + c(S)) + l((X,Y) * (S,W))}{2} -$$

$$\frac{(n_L(S) + n_L(W))|X| + (n_R(S) + n_R(W))|Y|}{2}$$

where we take the signal $+$ in $c(S)$ if $X_{|X|-m(X,Y)} = L$ and the signal $-$ otherwise.

Proof: The proof is analogous to the proof of the previous Proposition. ∎

Proposition 9 *Let (X,Y) and (S,W) be finite admissible pairs. Then, for each $n \in \mathbb{N}$, the genus of the Lorenz link associated to $(X,Y) * (S,W)^n$ is given by:*

$$g((X,Y) * (S,W)^n) =$$

$$\frac{1}{2}\left(\begin{array}{c} \left([c(X)c(Y)l(X,Y)]\,A_{33}^{n-1} + [c(S)c(W)l(S,W)]\sum_{i=0}^{n-2} a_i A_{33}^i\right) B_{13} \\ \\ +\alpha c(S,W) - [||X||Y||]\left[\begin{array}{cc} n_L(S) & n_L(W) \\ n_R(S) & n_R(W) \end{array}\right]^n \left[\begin{array}{c} 1 \\ 1 \end{array}\right] \end{array}\right),$$

where, considering the four cases of Remark 8, we have: in Case 1 $a_i = 1 = \alpha$ for all i; in Case 2 $a_i = (-1)^{i+n}$ and $\alpha = (-1)^{n+1}$; in Case 3 $a_i = -1 = \alpha$ for all i; in Case 4 $a_i = (-1)^{i+n+1}$ and $\alpha = (-1)^n$.

Proof: It is immediate, applying the formulas in Lemmas 5 and 6 in Equation 8. ∎

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